Gibbs Measures and Phase Transitions

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Chapter 0

Prerequisites

0.1 Lebesgue conditional expectation

Let $(X, \mathcal{X})$ be a measurable space, let $\mathcal{B}$ be a sub σ-algebra of $\mathcal{X}$.

Definition 0.1 (Lebesgue conditional expectation)**.** The **conditional expectation** of a $\mathcal{X}$-measurable function $f : X \to [0, \infty]$ is

$$\mu[f|\mathcal{B}] = ??$$

Lemma 0.2 (Characterisation of the Lebesgue conditional expectation)**.**

If $f : X \to [0, \infty]$ is a $\mathcal{X}$-measurable function, then $\mu[f|\mathcal{B}]$ is the μ-ae unique $\mathcal{B}$-measurable function $X \to [0, \infty]$ such that

$$\int_B \mu[f|\mathcal{B}] \; \partial\mu = \int_B f \; \partial\mu$$

for all $B \in \mathcal{B}$.

Proof. Standard machine. □

Chapter 1

Specifications of random fields

1.1 Preliminaries

Definition 1.1 (Juxtaposition)**.** Let E and S be sets. Let $\Delta \in \mathcal{P}(S)$, and let $\omega \in E^S$. We define

$$\mathrm{Juxt}_\omega : E^\Delta \to E^S \tag{1.1}$$
$$\zeta \mapsto \delta \mapsto \begin{cases} \zeta_\delta & \delta \in \Delta \\ \omega_\delta & \delta \notin \Delta \end{cases} \tag{1.2}$$

to be the **juxtaposition of** ζ **and** ω (for each $\zeta \in E^\Delta$).

Definition 1.2 (Cylinder events)**.** Let $(E, \mathcal{E})$ be a measurable space, and let S be a set. Then,

$$\mathcal{F} : \mathcal{P}(S) \to \{\text{sigma algebras on } E^S\} \tag{1.3}$$
$$\Delta \mapsto \sigma(\{\mathrm{proj}_\delta : E^S \to E \mid \delta \in \Delta\}) \tag{1.4}$$

defines the **cylinder events in** Δ (for each $\Delta \in \mathcal{P}(S)$), where each proj_δ is the coordinate projection at coordinate δ.

Definition 1.3 (Kernel)**.** Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. Then,

$$\mathrm{Ker}_{\mathcal{Y},\mathcal{X}} := \{\pi : \mathcal{X} \times Y \to [0, \infty] \mid \forall y \in Y, \pi(\cdot \mid y) \in \mathfrak{M}(X, \mathcal{X});\ \forall A \in \mathcal{X}, \pi(A \mid \cdot) \text{ is } \mathcal{Y}\text{-measurable}\}$$

defines the set of **kernels from** $\mathcal{Y}$ **to** $\mathcal{X}$, where $\mathfrak{M}(X, \mathcal{X})$ is the space of measures on X.

Definition 1.4 (Markov kernel)**.**
Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. We say that $\pi \in \mathrm{Ker}_{\mathcal{Y},\mathcal{X}}$ is a **Markov kernel** iff $\pi(X \mid \cdot) = 1$.

Let $(X, \mathcal{X})$ be a measurable space, let $\mathcal{B}$ be a sub σ-algebra of $\mathcal{X}$. Let $\pi \in \mathrm{Ker}_{\mathcal{B},\mathcal{X}}$.

Definition 1.5 (Proper kernel)**.**
π is **proper** iff $\pi(A \cap B \mid x) = \pi(A \mid x) \cdot \mathbf{1}_B(x)$ for all $A \in \mathcal{X}$, $B \in \mathcal{B}$ and $x \in X$.

Lemma 1.6 (Lebesgue integral characterisation of proper kernels)**.**
If π is proper, then

$$\int f(x)g(x)\ \pi(dx \mid x_0) = g(x_0) \int f(x)\ \pi(dx \mid x_0)$$

for all $x_0 \in X$ and functions $f, g : X \to [0, \infty]$ such that f is $\mathcal{X}$-measurable, g is $\mathcal{B}$-measurable.

Proof. Standard machine. ☐

Lemma 1.7 (Integral characterisation of proper kernels).

If π is a proper Markov kernel, then

$$\int f(x)g(x)\ \pi(dx \mid x_0) = g(x_0) \int f(x)\ \pi(dx \mid x_0)$$

for all $x_0 \in X$ and functions $f, g : X \to \mathbb{R}$ such that f is bounded $\mathcal{X}$-measurable and g is bounded $\mathcal{B}$-measurable.

Proof. Standard machine. ☐

Definition 1.8 (Conditional expectation kernel).

Let $\mu \in \mathfrak{M}(X, \mathcal{X})$. Then, $\pi \in \mathrm{Ker}_{\mathcal{B},\mathcal{X}}$ is a **conditional expectation kernel for** μ if $\mu(A \mid \mathcal{B}) = \pi(A \mid \cdot)$ μ-a.e.

Lemma 1.9 (Lebesgue integral characterisation of proper conditional expectation kernels).

If $\pi \in \mathrm{Ker}_{\mathcal{B},\mathcal{X}}$ is a conditional expectation kernel for μ, then

$$\mu[f \mid \mathcal{B}] = \int f(x)\ \pi(\partial x \mid \cdot)\ \mu\text{-a.e.}$$

for all $\mathcal{X}$-measurable functions $f : X \to [0, \infty]$.

Proof.

Standard machine. ☐

Lemma 1.10 (Integral characterisation of proper conditional expectation kernels).

If $\pi \in \mathrm{Ker}_{\mathcal{B},\mathcal{X}}$ is a conditional expectation kernel for μ, then

$$\mu(f \mid \mathcal{B}) = \int f(x)\ \pi(\partial x \mid \cdot)\ \mu\text{-a.e.}$$

for all bounded $\mathcal{X}$-measurable functions $f : X \to \mathbb{R}$.

Proof.

Standard machine. ☐

Lemma 1.11 (Characterisation of proper conditional expectation kernels, Remark 1.20).

Let $\mu \in \mathfrak{M}(X, \mathcal{X})$ be a finite measure and let $\pi \in \mathrm{Ker}_{\mathcal{B},\mathcal{X}}$ be a proper kernel. Then,

$$\pi \text{ is a conditional expectation kernel for } \mu \iff \mu\pi = \mu$$

Proof.

By the characterisation of conditional expectation,

$$\pi \text{ is a conditional expectation kernel for } \mu \iff \forall A \in \mathcal{X}, \forall B \in \mathcal{B}, \mu(A \cap B) = \int_B \pi(A|\cdot)\ \partial\mu$$

By properness of π,

$$\int_B \pi(A|\cdot)\ \partial\mu = \mu\pi(A \cap B)$$

Hence

$$\begin{aligned} \pi \text{ is a cond. exp. kernel with respect to } \mu &\iff \forall A \in \mathcal{X}, \forall B \in \mathcal{B}, \mu(A \cap B) = \mu\pi(A \cap B) && (1.5)\\ &\iff \forall A \in \mathcal{X}, \mu(A) = \mu\pi(A) && (1.6)\\ &\iff \mu = \mu\pi && (1.7) \end{aligned}$$

☐

1.2 Prescribing conditional probabilities

Definition 1.12 (Specification)**.**
A **specification** is a family of kernels γ : Finset $S \to \mathrm{Ker}_{\mathcal{F}_{S\setminus\Lambda},\mathcal{E}^S}$ which is **consistent**, in the sense that

$$\forall \Lambda_1, \Lambda_2 \in \mathrm{Finset}(S), \Lambda_1 \subseteq \Lambda_2 \implies \gamma_{\Lambda_1} \circ_k \gamma_{\Lambda_2} = \gamma_{\Lambda_2}$$

All specifications will be with parameter set S and state space $(E, \mathcal{E})$ in this chapter.

Definition 1.13 (Independent specification)**.**
A specification γ is **independent** iff

$$\forall \Lambda_1, \Lambda_2 \in \mathrm{Finset}(S), \gamma_{\Lambda_1} \circ_k \gamma_{\Lambda_2} = \gamma_{\Lambda_1 \cup \Lambda_2}$$

Definition 1.14 (Markov specification)**.**
A specification γ is a **Markov specification** iff γ_Λ is a probability kernel for every $\Lambda \in \mathrm{Finset}(S)$.

Definition 1.15 (Proper specification)**.**
A specification γ is **proper** iff the kernel γ_Λ is proper for every $\Lambda \in \mathrm{Finset}(S)$.

Definition 1.16 (Gibbs measures)**.** Given a specification γ, a **Gibbs measures specified by** γ is a measure $\nu \in \mathfrak{M}(E^S, \mathcal{E}^S)$ such that $\gamma_\Lambda(A|\cdot)$ is a conditional expectation kernel for ν for all $A \in \mathcal{E}^S$ and $\Lambda \in \mathrm{Finset}(S)$.

Lemma 1.17 (Characterisation of Gibbs measures, Remark 1.24)**.**
Let γ be a *proper* specification with parameter set S and state space $(E, \mathcal{E})$, and let $\nu \in \mathfrak{P}(E^S, \mathcal{E}^S)$. TFAE:

1. $\nu \in \mathcal{G}(\gamma)$.
2. $\gamma_\Lambda \circ_m \nu = \nu$ for all $\Lambda \in \mathrm{Finset}(S)$.
3. $\gamma_\Lambda \circ_m \nu = \nu$ frequently as $\Lambda \to S$.

Proof.
1 is equivalent to 2 by Lemma 1.9. 2 trivially implies 3. Now, 3 implies 2 because for each Λ there exists some $\Lambda' \supseteq \Lambda$ such that $\gamma_{\Lambda'} \circ_k \nu = \nu$. Then

$$\nu\gamma_\Lambda = \nu\gamma_{\Lambda'}\gamma_\Lambda = \nu\gamma_{\Lambda'} = \nu$$

□

1.3 λ-specifications

Let S be a set, $(E, \mathcal{E})$ be a measurable space and ν a measure on E.

Definition 1.18 (Product probability measure)**.** Let I be a set. Suppose for each $i \in I$ that $(\Omega_i, \mathcal{B}_i, P_i)$ is a probability space. Then, $P := \bigotimes_{i \in I} P_i$ is a well-defined product probability measure on $\prod_{i \in I} \Omega_i$.

Definition 1.19 (Independent Specification with Single Spin Distribution (ISSSD)).
The **Independent Specification with Single Spin Distribution** ν is

$$\text{ISSSD} : \mathfrak{P}(E, \mathcal{E}) \to \text{Finset}(S) \to \mathcal{E}^S \times E^S \to \overline{\mathbb{R}_{\geq 0}} \quad (1.8)$$

$$\nu \mapsto \Lambda \mapsto (A \mid \omega) \mapsto \left(\nu^\Lambda \left(\text{Juxt}_\omega^{-1}(A)\right)\right) \quad (1.9)$$

defines the **Independent Specification with Single Spin Distribution with** ν (for each $\nu \in \mathfrak{P}(E, \mathcal{E})$), where ν^Λ is the usual product measure.

Lemma 1.20 (Independence of ISSSDs).
ISSSD(ν) is independent.

Proof. Immediate. □

Definition 1.21 (ISSSDs are specifications).
ISSSD(ν) is a specification.

Lemma 1.22 (ISSSDs are proper specifications).
ISSSD(ν) is a proper specification.

Proof.
We already know it's a specification. Properness is immediate. □

Lemma 1.23 (Uniqueness of a Gibbs measure specified by an ISSSD).
There is at most one Gibbs measure specified by ISSSD(ν).

Proof.
See book. □

Lemma 1.24 (Existence of a Gibbs measure specified by an ISSSD).
The product measure ν^S is a Gibbs measure specified by ISSSD(ν).

Proof. Immediate. □

Definition 1.25 (Modifier).
A **modifier of** γ is a family

$$\rho : \text{Finset}(S) \to \Omega \to [0, \infty[$$

such that the corresponding family of kernels $\rho\gamma$ is a specification.

Lemma 1.26 (Modifier of a modifier).
Modifying a specification γ by ρ_1 then ρ_2 is the same as modifying it by their product.

Proof. TODO □

Lemma 1.27 (A modifier of a proper specification is proper).
If γ is a specification and ρ a modifier of γ, then $\rho\gamma$ is a proper specification.

Proof.
For all $\Lambda \in \text{Finset}(S)$, $A \in \mathcal{E}^S$, $B \in \mathcal{F}_{S \setminus \Lambda}$, $\eta : S \to E$, we want to prove

$$(\rho\gamma)_\Lambda(A \cap B|\eta) = 1_B(\eta)(\rho\gamma)_\Lambda(AB|\eta)$$

Expanding out, this is equivalent to

$$\int_{\zeta \in A \cap B} \rho_\Lambda(\zeta)\ d(\gamma_\Lambda(\eta)) = 1_B(\eta) \int_{\zeta \in A} \rho_\Lambda(\zeta)\ d(\gamma_\Lambda(\eta))$$

which is true by Lemma 1.6 with $f = 1_A \rho_\Lambda$, $g = 1_B$. □

Lemma 1.28 (Every specification is a modification of some ISSSD, Remark 1.28.5)**.**
If E is countable, ν is the counting measure and γ is any specification, then

$$\rho_\Lambda(\eta) = \gamma_\Lambda(\{\sigma_\Lambda = \eta_\Lambda\}|\eta)$$

is a modifier from ISSSD(ν) to γ.

Proof. For all $\Lambda \in \mathrm{Finset}(S)$, A measurable, $\eta : S \to E$, we have

$$(\rho\,\mathrm{ISSSD}(\nu))_\Lambda(A|\eta) = \int_\zeta \rho_\Lambda(\zeta)\,\mathrm{ISSSD}(\nu)(d\zeta|\eta) \tag{1.10}$$

$$= \int_\zeta \gamma_\Lambda(\{\sigma_\Lambda = \eta_\Lambda\}|\eta)\,\mathrm{ISSSD}(\nu)(d\zeta|\eta) \tag{1.11}$$

$$= \gamma_\Lambda(A|\eta) \tag{1.12}$$

□

Proposition 1.29 (Characterisation of modifiers, Proposition 1.30.1)**.**
If ρ is a family of measurable densities and γ is a proper specification, then TFAE

1. ρ is a modifier of γ
2. For all Λ_1, Λ_2 with $\Lambda_1 \subseteq \Lambda_2$ and all $\eta : S \to E$, we have

$$\rho_{\Lambda_2} = \rho_{\Lambda_1} \cdot (\gamma_{\Lambda_1}\rho_{\Lambda_2}) \quad \gamma_{\Lambda_2}(\cdot|\eta)\text{-a.e.}$$

Proof.
- ($\implies$) $\rho_{\Lambda_2} = \rho_{\Lambda_1} \cdot (\gamma_{\Lambda_1}\rho_{\Lambda_2}) \quad \gamma_{\Lambda_2}(\cdot|\eta)$-a.e.
  - $\implies \rho_{\Lambda_2}\gamma_{\Lambda_2} =$

□

Proposition 1.30 (Characterisation of modifiers of independent specifications, Proposition 1.30.2)**.**
If ρ is a family of measurable densities and γ is a proper independent specification, then TFAE

1. ρ is a modifier of γ
2. For all Λ_1, Λ_2 with $\Lambda_1 \subseteq \Lambda_2$, $\eta : S \to E$ and $\gamma_{\Lambda_2 \setminus \Lambda_1}(\cdot|\alpha)$-almost all $\eta_2 : S \to E$, we have

$$\rho_{\Lambda_2}(\zeta_1)\rho_{\Lambda_1}(\zeta_2) = \rho_{\Lambda_2}(\zeta_2)\rho_{\Lambda_1}(\zeta_1)$$

   for $\gamma_{\Lambda_1}(\cdot|\eta_2) \times \gamma_{\Lambda_2}(\cdot|\eta_2)$-almost all (ζ_1, ζ_2).

Proof. □

Definition 1.31 (Premodifier, Definition 1.31)**.** A family of measurable functions $h_\Lambda : (S \to E) \to [0, \infty[$ is a **premodifier** if

$$h_{\Lambda_2}(\zeta)h_{\Lambda_1}(\eta) = h_{\Lambda_1}(\zeta)h_{\Lambda_2}(\eta)$$

for all $\Lambda_1 \subseteq \Lambda_2$ and all $\zeta, \eta : S \to E$ such that $\zeta_{\Lambda_1^c} = \eta_{\Lambda_1^c}$.

Lemma 1.32 (Modifiers are premodifiers)**.** If ρ is a modifier of $\text{ISSSD}(\nu^S)$, then it is a premodifier if any of the following conditions hold:

1. E is countable and ν is equivalent to the counting measure.
2. E is a second countable Borel space.
3. ν is everywhere dense.
4. For all $\Lambda_1 \subseteq \Lambda_2$ and all $\eta : S \to E$, $\zeta \mapsto \rho_{\Lambda_1}(\zeta\eta_{\Lambda_1^c})$ is continuous on E^{Λ_1}.

Proof.

1. Use Proposition **??**.
2. Omitted.
3. Omitted.
4. Omitted.

□

Lemma 1.33 (Premodifiers give rise to modifiers, Remark 1.32)**.** If h is a premodifier and ν is such that $0 < \nu_\Lambda h_\Lambda < \infty$ for all Λ, then

$$\rho_\Lambda := \frac{h_\Lambda}{\text{ISSSD}(\nu)_\Lambda h_\Lambda}$$

is a modifier of $\text{ISSSD}(\nu)$.

Proof.
TODO □

Chapter 2

Gibbsian specifications

2.1 Potentials